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DEPARTMENT OF ECONOMETRICS

PARTIAL EQUILIBRIUM IN A MARKET IN THE CASE OF INCREASING RETURNS AND SELLING COSTS

by

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Introduction. [1])

In most models of general equilibrium, the technology of producers is represented by convex production sets (e.g. Debreu [2]), which imply increasing marginal cost functions. Convex production sets or increasing marginal cost functions permit the definition of supply functions (or correspondences) and thus an equilibrium may be defined by equating supply and demand.

In partial analysis, both for competitive and monopolistic markets, it is often assumed that marginal cost functions decrease first and then increase ("U shaped curves"), as e.g. in Chamberlain [1] and Shubik [9]. Here the solution always lies in the increasing part of the marginal cost function, a case also considered in equilibrium theory by Debreu [3] [2]).

If however there are increasing returns to scale for any level of output, so the production sets are not convex and the cost functions that they imply show decreasing mean cost for any positive value of output, then supply functions are not defined, as was already noted by Sraffa [10] (p. 543) and Viner [12], since the profit function does not attain a finite non zero maximum. So equality of supply and demand cannot be a basis of equilibrium.

One way out is to assume that firms produce differentiated products with high but finite elasticities of substitution (see Sraffa [10]). Now each firm is a monopolist for his own product competing with firms producing close substitutes.

In the present paper however, we construct a partial equilibrium

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1) Part of the research for this paper was done while the author was a researchfellow at CORE, Louvain. The author thanks J. Dalmulder and P. Ruys for their comments.

2) Just before I finished the present paper, I received the book of Marshak and Selten [5], who in chapter IV consider a general equilibrium model with non decreasing returns to scale. Their approach is similar to the one in the present paper for the case without selling costs, since they introduce fixed market shares.

model of a market where all firms have decreasing mean cost functions and produce a single homogeneous commodity: consumers consider the products of all firms equivalent.
In this case any producer, who produces at all, will try to sell as much as possible. Therefore it is necessary to distribute total demand over producers. In order to do this the demand schedule is completed by adding a distribution of market shares to the traditional demand function. We consider two cases: in the first the shares are given data for each producer, in the second they may be influenced by selling activity. The markets considerd in this paper are essentially n-persons non zero sum games. The equilibrium concept which is used is the game theoretical concept of Nash Equilibrium which is a non cooperative solution. It is assumed here that firms do not consider the effect of their behaviour on their competitors (apart from the assumption that they do expect their competitors to follow any price decrease). This may only be plausible if the number of firms is "large" (whatever that means). There certainly exist other equilibrium concepts (cooperative solutions) which might be interesting in the present case. However the Nash equilibrium approach keeps the analysis nearest to competitive behaviour of firms with convex production sets.
But of course the market has also important features of monopolistic competition since each frim faces a decreasing demand function. (see Samuelson [6])

It should be pointed out, that the solutions in our market are not efficient, unless there is only a single firm: if there are increasing returns to scale the only efficient way to produce is to have produced all output of a commodity by one firm. Papers by Scarf [7] and Dierker, Fourgeaud and Neufeind [4] consider general equilibrium solutions in an economy with increasing returns, which guarantee efficiency and which are enforced by some planning mechanism; in these papers it is assumed that each commodity is produced by one firm. A second source of inefficiency are the selling costs, which are, within the framework of the model below, nothing but an expensive method to distribute total demand over producers.

In the first part of the present paper we consider production and consumption and introduce the concepts necessary in part II, where the markets are analysed.

PART I CONCEPTS

2. PRODUCERS.

Let $N = \{1,2,...,n\}$ be the set of firms that can produce a certain homogeneous commodity and assume that they all produce a single commodity using several inputs. Each firm's technology can be represented by a production set $Y_i \subset R^{\ell+1}$; a typical element of this set being a vector (y_i, z_i), where y_i is the quantity produced by i and $z_i = (z_i^1, z_i^2, ..., z_i^{\ell})$ is the vector of imputs used to produce y_i, hence $z \leqq 0$.
Let $(p,q) \in R_+^{\ell+1}$ be a price vector, p being the price of output and q being the vector of input prices. The number $\pi_i = py_i + qz_i$ is firm i's profit.
We define i's cost function: $f_i: R^{\ell+1} \to R^1$, where $f_i(y,q)$ is the minimal cost to produce y at input prices q:

$$f_i(y,q) = \min \{- c \mid c = qz \text{ and } (y,z) \in Y_i\}$$

The following assumptions are standard.
For all $i \in N$:

1 $0 \in Y_i$

2 $Y_i \cap R_+^{\ell+1} = \{0\}$

3 $R_-^{\ell+1} \subset Y_i$

4 Y_i is closed

5 $(y,z) \in Y_i$ and $(y',z') \leqq (y,z) \Rightarrow (y'z') \in Y_i$

From these assumptions it directly follows that the cost function is non-decreasing both in y and q:

$$y \geqq y' \Rightarrow f_i(y,q) \geqq f_i(y',q)$$

$$q \geqq q' \Rightarrow f_i(y,q) \geqq f_i(y,q')$$

We call <u>profitable sales correspondence</u> the correspondence $H_i(p,q)$, that associates to each price vector the quantities y of output which are profitable, i.e. no loss is made if they are sold

$$H_i(p,q) = \{y|\exists z: (y,z) \in Y_i \text{ and } py + qz \geqq 0\}$$

$$= \{y|py - f_i(y,q) \geqq 0\} = \{y|p \geqq \frac{f_i(y,q)}{y}\}$$

A firm will only operate if its sales are in $H_i(p,q)$.

3. CONVFX PRODUCTION SETS.

If we assume also
<u>assumption C</u>: $\forall_i: Y_i$ is <u>convex</u>, then it is well known that for a firm maximizing profits a <u>supply correspondence</u> exists.
Let $\pi_i(p,q) = \max \{py+qz|(y,z) \in Y_i\}$ be i's profit function.
$\pi_i(p,q)$ is non negative and finite in some closed set $Q \in R^{\ell+1}$.
The supply correspondence $b_i: R^{\ell+1} \to R^{\ell+1}$:

$$b_i(p,q) = \{y,z|py+qz = \pi_i(p,q) \text{ and } (y,z) \in Y_i\}$$

is non empty, closed and convex valued in Q and it is upper hemi continuous and compact valued in Int Q.
The first component $b_i^0(p,q)$ represents i's supply of output at prices (p,q) and for $k = 1,2,\ldots,\ell$, $-b_i^k(p,q)$ is i's demand for inputs. It is non-decreasing in p, for fixed q, in its first component.

$$p \geqq p' \Rightarrow b_i^0(p,\bar{q}) \geqq b_i^0(p',\bar{q})$$

The total supply correspondence $b(p,q) = \sum_i b_i(p,q)$ is the aggregate supply of all firms and has the same properties.
The cost function $f_i(y,q)$ is convex in y:

$$\lambda f_i(y,q) + (1-\lambda)f_i(y'q) \geqq f_i(\lambda y + (1-\lambda)y',q)$$

and we have

$$\pi_i(p,q) = \max_{y \geqq 0} [qy - f_i(y,q)]$$

Further the profitable output correspondence contains the supply correspondence

$$H_i(p,q) \supset b_i^0(p,q)$$

and $H_i(p,q)$ is a convex set.
For fixed $\bar{q}$, $H_i(p,\bar{q}) \subset H_i(p,\bar{q})$, if $p > p'$; $H_i(p,q)$ is compact (an interval) if $b_i(p,q)$ is compact.

4. INCREASING RETURNS.

In what follows we shall not assume convexity, but we consider the case of increasing returns to scale. Therefore, instead of assumption C, we make the following two assumptions:

6 For all $y \geqq 0$: $\{z \mid (y,z) \in Y_i\}$ is convex

7 For all $\lambda > 1$: $(y,z) \in Y_i \Rightarrow \exists y'$: $y' > \lambda y$ and $(y',\lambda z) \in Y_i$

Assumption 6 ensures that the set of inputs, which permit production of y, is convex. By assumption 7 a proportional extension of inputs leads to a more than propertional extension of output. Now a reasonable supply correspondence is no longer defined:
for any $(p,q) \in R^{\ell+1}$, we have either

$$\pi_i(p,q) = \max \{py+qz \mid (y,z) \in Y_i\} = 0$$

or

$$\pi_i(p,q) = \max \{py+qz \mid (y,z) \in Y_i\} = \sim$$

This follows from assumption 7: suppose $py+qz = \alpha > 0$, $\alpha < \sim$ and $(y,z) \in Y_i$; then $\lambda(y,z) \in \text{Int } Y_i$, hence $p(\lambda y) + q(\lambda z) = = \lambda\alpha$; so α can never be a maximum. This implies that supply

<u>is either zero or infinite</u>. So equality of supply and demand in the traditional sense cannot be the basis of an equilibrium. Because of assumption 6, the cost function exists and also a demand correspondence for inputs depending on y and q. The cost function is increasing in y, for $\bar{q}$ fixed:

$$f_i(y,\bar{q}) > f_i(y',\bar{q}) \quad \text{if} \quad y > y'$$

From assumption 7 it follows that the mean cost function is decreasing, for $\bar{q} > 0$ fixed:

$$\frac{f_i(y,\bar{q})}{y} < \frac{f_i(y',\bar{q})}{y'} \quad \text{if} \quad y > y'$$

It is easy to prove, that if f is differentiable, then

$$\frac{\partial f_i(y,\bar{q})}{\partial y} < \frac{f_i(y,\bar{q})}{y} \quad \text{for all } y$$

The profitable sales correspondence is well defined for all p,q. For $\bar{q}$ fixed and p sufficiently low, that is if $\frac{f_i(y,\bar{q})}{y} < p$, for all y, then $H_i(p,\bar{q}) = 0$. For larger values of p, $H_i(p,\bar{q})$ is a convex set bounded below and <u>unbounded above</u>.
The lower bound of the profitable sales correspondence, we call

<u>minimum sales function</u>:

$$h_i(p,\bar{q}) = \min \{y \mid y \in H_i(p,\bar{q})\}$$

This function is the inverse of the mean cost function:

$$y = h_i(p,q) \Leftrightarrow p = \frac{f_i(y,q)}{y}$$

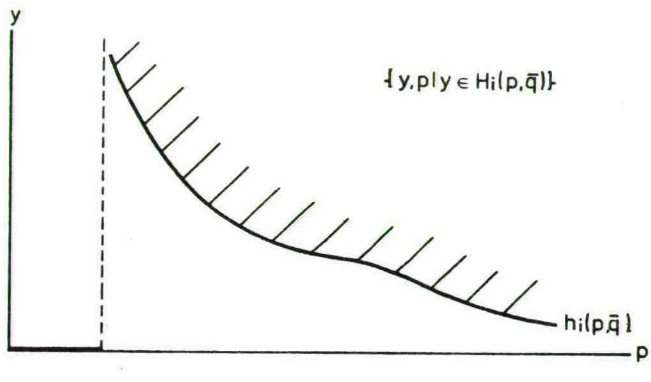


Figure 1

It is decreasing in p.

A pair (y,p) is feasible if $y \in H_i(p,q)$, that is if the y sold is at least $h_i(p,q)$. If p and q are given, then a firm will only produce a quantity y if

1) he can sell y at price p

2) $y \in H_i(p,q)$

The firm is willing to sell any quantity of product larger than $h_i(p,q)$: the more he sells at this price, the higher will be his profit. Since he cannot hope to sell an infinite quantity, his sales are lower than he would like them to be; so the problem is, how much can each firm sell at a price p?

Remark

Compare the shape of the profitable sales correspondence in figure 1, to the shape of this correspondence if the production set is strictly convex, as is shown in figure 2. Both $H_i(p,\bar{q})$ and the supply function converge to 0, if p converges to some minimum price $\bar{p}$. The minimum sales function is always 0. Sales will never be larger than $b_i(p,\bar{q})$!

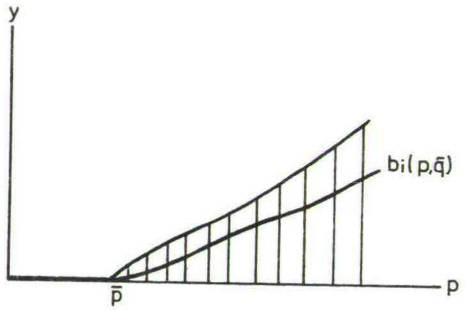


Figure 2

5. CONSUMERS AND PRODUCERS.

The "traditional" theory of consumer choice is only concerned with the question how much a consumer demands and not with the question from which producer he buys. This approach seems to be satisfactory if the commodity is produced with decreasing returns. Assume first that the production sets are strictly convex. Then the supply correspondence is single valued. So at each price vector the producer offers some quantity y_i, not more and not less, so total supply equals $\sum_i y_i$. If preferences are also strictly convex, then demand x_j of each consumer and total demand $\sum_j x_j$ are also single valued and at an equilibrium price we have by definition $\Sigma x_j(p) = \Sigma y_i(p)$. At this equilibrium price each consumer will be able, possibly after some search, to find a producer who is willing to sell the required quantity. The consumer might first address to a firm which has already sold out, then he will look for another one. Each producer will be able to find consumers who are willing to buy and the market will clear. Things are slightly more complex if production sets or preferences are convex but not strictly convex, but not essentially different.
If however all production sets are as described in the previous section, so that increasing returns appear at any level of production, then things become completely different. At a given

price vector each producer has first to decide if he will produce at all. He will not produce if the cannot sell at least his minimum sales $h_i(p,q)$. If he decides to produce, he will try to sell as much as possible. So if a consumer directs himself to such a producer, he will always be served. Therefore it becomes necessary to determine how consumers choose their seller, i.e. to which firm they address first. We want to keep the theory as simple as possible and similar to the theory with decreasing returns. Therefore we assume that each consumer determines the quantity demanded only by considering prices and not by considering sellers, so that traditional demand theory remains the basis of individual and total demand. This implies that each consumer will determine his demand by considering the price raised by the cheapest producer and he will only buy from a producer raising this lowest price. Hence all producers must ask the same price; a more expensive producer will not sell anything.
This also implies that any producer has the possibility to lower the market price. His competitors will have to follow him, if not they loose all their sales. Increasing the price however is only possible by cooperative action of all sellers. Hence each individual firm faces a "kinked demand function" (see [11]).
The consumers should choose their seller among the producers asking the same price. Different assumptions about their behaviour are possible, e.g.

- The consumer has no preference at all for any seller. Buying from i or j is a choice between indifferent alternatives. So the choice will be random or guided by some conscient or inconscient mechanism.

- The consumer has certain preferences for producers, but these preferences are lexicographically related to the preferences for commodities; his preferences among sellers are so to say "second order preferences", or to make this more precise: if x_{kj} is the quantity of the k-th commodity bought from the j-th firm, then there are three preference relations:

$\underset{\sim 1}{\succ}$ amoung bundles of the type $(\sum_j x_{kj})$

$\underset{\sim 2}{\succ}$ amoung bundles of the type (x_{kj})

and an overall preference relation $\underset{\sim}{\succ}$, such that

$$x \succ x' \Leftrightarrow x \succ_1 x', \quad \text{or} \quad x \sim_1 x' \quad \text{and} \quad x \succ_2 x'$$

Both approaches leave open the possibility that the consumer's decisions with respect to sellers can be influenced by selling activities of the producers, as advertising, the sending around of salesman, giving premiums to shopkeepers, etc. It seems reasonable to assume that consumers may be easily persuaded to make a particular choice, if this choice is considered to be equivalent to other alternatives, and that a (weak) preference for one firm over another could be easily inverted.
We shall not explore this question further with respect to individual consumers, but attack the question more globally and consider only the aggregate behaviour of consumers.
Therefore we introduce the concept of the market share of firm i. The market share ρ_i is the fraction of total demand $x(p,q)$, where $x(p,q)$ is the traditional demand function, that is addressed to firm i. So i's sales y_i are equal to $\rho_i x(p,q)$.
The fraction ρ_i may be depend on any variable in the economy.
We shall assume that it only depends on two factors:

- the producers who are in the market
- the selling activity of each producer in the market.

6. MARKET SHARE DISTRIBUTIONS.

$N = \{1,2,\dots,n\}$ is the set of firms that can produce a certain commodity with increasing returns to scale.
We first consider the case where selling costs do not exist.
So the firms have only to decide to produce or not. Let $a \in \{0,1\}^n$ be the vector of activity indices, with components

0 and 1. $a_i = 1$ means that firm 1 will produce and consequently sell as much as possible at the going price p (he is an active producer) and $a_i = 0$ means that i does not produce (he is sleeping).

Notation.

We define $a = (a_1, a_2, \ldots, a_n) = (a_i, \tilde{a}_i)$, where $\tilde{a}_i = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, so $\tilde{a}_i$ is the vector of activity indices of all firms, except i.

Definition 6.1

If there is no selling activity possible, a market share distribution is a mapping $\rho\colon \{0,1\}^n \to [0,1]^n \subset R_+^n$, where $\rho_i(a)$ is i's market share and

(1) $\quad 0 \leqq \rho_i(a) \leqq 1$

(2) $\quad \rho_i(a) = 0$ if $a_i = 0$

(3) $\quad \Sigma\rho_i(a) = 1$ if $a \neq 0$

We assume for this case that the share of each active firm (strictly) decreases if the set of active firms increases, i.e. if a new active firm enters:

ASSUMPTION M 0 $\quad \rho_i(1, \tilde{a}_i') < \rho_i(1, \tilde{a}_i)$ if $\tilde{a}_i' \geq \tilde{a}_i$

An example of such a distribution is

$$\rho_i(a) = \frac{a_i}{\Sigma a_j}$$

where all active producers have the same share, e.g. because there are many consumers who choose randomly [1].

[1] Marshak and Selten [5] use the function $\rho_i = \frac{\alpha_i}{\Sigma\alpha_i}$, where the α_i are given positive numbers.

Next we consider the case where selling costs are possible. Let s_i denote firm i's selling expenses and $v_i = (a_i, s_i) \in V_i = \{0,1\} \times R_+$ denotes the combination of i's activity index and his selling expenses (obviously $a_i = 0$ and $s_i > 0$ which is not excluded formally, cannot be a rational decision). The vector $v \in (\{0,1\} \times R_+)^n$ denotes the decision taken by all n firms.

Notation.

We define $v = (v_1, v_2, \dots, v_n) = (v_i, \tilde{v}_i)$, where by $\tilde{v}_i = (v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$. We also write $v = (a_i, s_i, \tilde{v}_i) = (a_i, s_i, \tilde{a}_i, \tilde{s}_i)$.

Definition 6.2

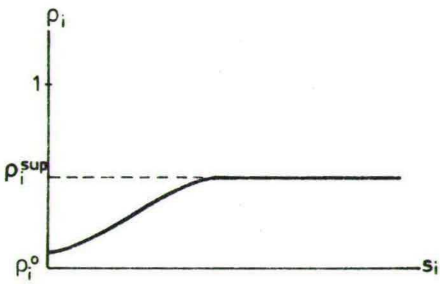
If there are selling activities, a market share distribution is a mapping $\rho: V \to [0,1]^n$, where $V = (\{0,1\} \times R_+)^n$, such that

(1) $0 \leqq \rho_i(v) \leqq 1$

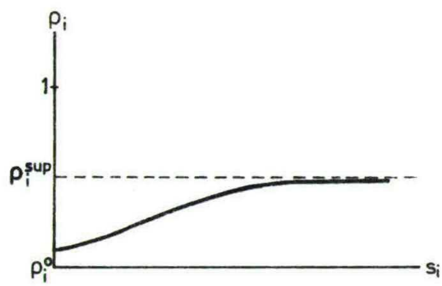
(2) $\rho_i(v) = 0$ if $a_i = 0$

(3) $\Sigma \rho_i(v) = 1$ if $a \neq 0$

We assume: (M1) The share of each active firm does not increase if the set of active firms increases (it is not assumed that it increases, for, if the j-th firm becomes active, without having a positive amount of selling costs, it does not necessarily get a non-zero share). (M2) The share of firm i does not increase if the j-th firm increases its selling expenses. (M3) Selling costs are effective below a certain level, i.e. by an increase of its selling expenses a firm increases its market share at least within some interval, but it remains possible that at some level an increase is not effective. (M4) Requires that if this occurs, there is a satiation level of selling costs: below this level an increase is effective, but not above this level. This is illustrated in figures 3a and 3b. (M5) Finally it is assumed that the market share is twice differentiable (hence continuous) with respect to the selling expenses of the firm itself and the ones of his competitors.



<u>Figure 3a</u>



<u>Figure 3b</u>

<u>Assumptions</u>

M 1 $\quad \rho_i(1,s_i,\tilde{a}_i,\tilde{s}_i) \geqq \rho_i(1,s_i,\tilde{a}_i',\tilde{s}_i)$ if $\tilde{a}_i \leqq \tilde{a}_i'$

M 2 $\quad \rho_i(1,s_i,\tilde{a}_i,\tilde{s}_i) \geqq \rho_i(1,s_i,\tilde{a}_i,\tilde{s}_i')$ if $\tilde{s}_i \leqq \tilde{s}_i'$

M 3 $\quad$ For any $\tilde{v}_i$ there exist s_i and s_i', where $s_i > s_i'$, such that $\rho_i(1,s_i,\tilde{v}_i) > \rho_i(1,s_i',\tilde{v}_i)$

M 4 $\quad$ If $\rho_i(1,s_i,\tilde{v}_i) = \rho_i(1,s_i',\tilde{v}_i)$ and $s_i < s_i'$, then for all $s_i'' \geqq s_i'$

$$\rho_i(1,s_i'',\tilde{v}_i) = \rho_i(1,s_i',\tilde{v}_i)$$

M 5 $\quad \rho_i(1,s_i,\tilde{a}_i,\tilde{s}_i)$ is twice differentiable with respect to s_i and s_j $(j \neq i)$.

An example of such a distribution is

$$\rho_i(a,s) = \frac{s_i}{\Sigma s_j}$$

In this case a does not explicitely occur: if $s_i = 0$, the market share is 0 and hence $a_i = 0$.

A similar function is used by Schmalensee [8] for market shares with respect to advertising.

Remark

In the same way as $f_i(y,q)$ is the cost function, assuming an optimal choice of technology, the amount s_i is assumed to be allocated in such a way among different selling activities, that its effect on ρ_i is maximal.

Let for given values of $\tilde{v}_i$

$$\rho_i^0(\tilde{v}_i) = \rho_i(1,0,\tilde{v}_i) = \min\{\rho_i(1,s_i,\tilde{v}_i) \mid s_i \geqq 0\}$$

and

$$\rho_i^{\sup}(\tilde{v}_i) = \sup\{\rho_i(1,s_i,\tilde{v}_i \mid s_i \geqq 0\}$$

$\rho_i^{\sup}(\tilde{v}_i)$ exists since the market shares are non-negative and bounded above. The share attains its minimum at $s_i = 0$, because of assumptions M 3 and M 5. If there does not exist a satiation level for selling costs, then the market share is increasing for all positive s_i and this implies that the supremum is not attained (see fig. 3b), hence if $s_i \to \sim$, then $\rho_i(1,s_i,\tilde{v}_i) \to \rho_i^{\sup}(\tilde{v}_i)$. If there exists a satiation level, then the supremum is attained for any amount of selling expenses above the satiation level (see fig. 3a). So we have

Property 6.3

(1) If $\frac{\partial}{\partial s_i} \rho_i(1,s_i,\tilde{v}_i) > 0$ for all $s_i \geqq 0$, then

$$s_i \to \sim \Rightarrow \rho_i(1,s_i,\tilde{v}_i) \to \rho_i^{\sup}(\tilde{v}_i)$$

(2) If there exists s_i such that $\frac{\partial}{\partial s_i} \rho_i(1,s_i,\tilde{v}_i) = 0$, then

$$\rho_i(1,s_i',\tilde{v}_i) = \rho_i^{\sup}(\tilde{v}_i) \text{ for all } s_i' \geqq s_i$$

For given values of $\tilde{v}_i$, the selling costs can be expressed as a function of firm i's market share.

The function $g_i(\cdot,\tilde{v}_i)$ maps the interval $\{\rho_i | 0 \leqq \rho_i \leqq \rho_i^{sup}(\tilde{v}_i)\}$ into R_+:

$$g_i(\rho_i,\tilde{v}_i) = \min \{s_i | \rho_i(1,s_i,\tilde{v}_i) \geqq \rho_i\}$$

Since the market share is an increasing function of the selling costs, for s_i positive and below the satiation level, the selling cost function g_i is also increasing for $\rho_i(\tilde{v}_i) \leqq \rho_i \leqq \rho_i^{sup}(\tilde{v}_i)$ and has value zero for all $\rho_i \leqq \rho_i^0(\tilde{v}_i)$. If a satiation level exists, then g_i attains this level at $\rho_i^{sup}(\tilde{v}_i)$ if not then g_i increases indefinitely if ρ_i converges to $\rho_i^{sup}(\tilde{v}_i)$ (see figures 4a and 4b).

So we have

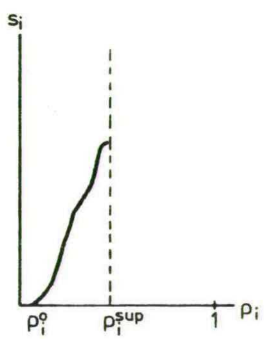


Figure 4a

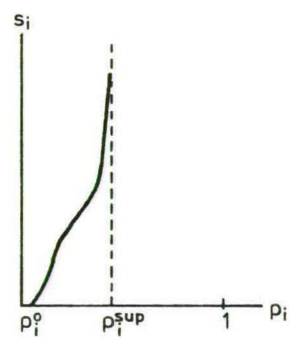


Figure 4b

Property 6.4

(1) $g_i(\rho_i,\tilde{v}_i) = 0$ if $\rho_i \leqq \rho_i^0(\tilde{v}_i)$

(2) $g_i(\rho_i,\tilde{v}_i)$ is strictly increasing if $\rho_i^0(\tilde{v}_i) \leqq \rho_i \leqq \rho_i^{sup}(\tilde{v}_i)$

(3) g_i is twice differentiable if $\rho_i^0(\tilde{v}_i) \leqq \rho_i \leqq \rho_i^{sup}(\tilde{v}_i)$

(4) if $\rho_i^{sup}(\tilde{v}_i) = \max_{s_i} \rho_i(1,s_i,\tilde{v}_i)$, then $g_i(\rho_i,\tilde{v}_i) < \infty$

(5) otherwise $\rho_i \to \rho_i^{sup}(\tilde{v}_i) \Rightarrow g_i(\rho_i,\tilde{v}_i) \to \infty$

PART II NASH EQUILIBRIA IN A MARKET FOR A COMMODITY PRODUCED WITH INCREASING RETURNS.

7. ASSUMPTIONS.

$N = \{1,2,\ldots,n\}$ is the set of firms that can produce a certain commodity. Prices q of all other commodities in the economy are kept fixed.
The cost function of the i'th firm is $f_i(y)$ (instead of $f_i(y,q)$).
Demand is given by:

- a total demand function $x(p)$ (instead of $x(p,q)$)
- a market share distribution $\rho_i(v)$.

The cost functions $f_i(y)$ are assumed to be increasing, continuous and twice differentiable for $y > 0$, both mean and marginal costs are decreasing and mean costs converge to some positive number c_i.

Assumptions on $f_i(y)$, (for all $i \in N$)

C 1 $\quad y > 0 \Rightarrow f_i(y) > 0;\ f_i(0) = 0$

C 2 $\quad y > y' \Rightarrow f_i(y) > f_i(y')$

C 3 $\quad f_i$ is twice differentiable for $y > 0$ (hence continuous)

C 4 $\quad y > y' \Rightarrow \dfrac{f_i(y)}{y} < \dfrac{f_i(y')}{y'}$

C 5 $\quad y > y' \Rightarrow \dfrac{df_i(y)}{dy} < \dfrac{df_i(y')}{dy}$

C 6 $\quad$ there exist $c_i > 0$, such that, for all y

$$\frac{f_i(y)}{y} > c_i \text{ and } y \to \infty \Rightarrow \frac{f_i(y)}{y} \to c_i$$

The demand function is assumed to be of a classical type: decreasing, twice differentiable and such that total revenue $px(p)$ first increases with p, then attains a single maximum and then decreases.

Assumptions on $x(p)$

D 1 $\quad p > 0 \Rightarrow x(p) \geqq 0$

D 2 $\quad p > p' \Rightarrow x(p) < x(p')$

D 3 $\quad x(p)$ is twice differentiable for $p > 0$ (hence continuous)

D 4 $\quad$ there exists $p^0 > 0$, such that: $p \neq p^0 \Rightarrow p^0 x(p^0) > px(p)$

$$p < p^0 \Rightarrow d\,\frac{px(p)}{dp} > 0 \text{ and } p > p^0 \Rightarrow d\,\frac{px(p)}{dp} < 0$$

Further we assume that for high prices production is not profitable for any firm, because demand becomes too low.

Assumption E There exist a price r, such that $x(r) > 0$ for all i: $p > r \Rightarrow px(p) < f_i(x(p))$ or $x(p) = 0$.

The assumptions on the market share distributions are given in section 6.

8. MONOPOLY PRICES

We define the set P as the set of prices such that demand is positive:

$$P = \{p \mid x(p) > 0\}$$

By assumption D 2, P is an interval.
Further P_i is defined to be the set of prices at which there exists some profitable non-zero output for firm i:

$$P_i = \{p \mid H(p) \neq \{0\}\} = \{p \mid p > c_i\}$$

the equality following from assumption C 6. It proves that P_i

is an open half line. P_i is independent of the demand function. Given the demand function, the set of profitable prices of firm i at share $\rho_i > 0$, is defined

$$P_i(\rho_i) = \{p \mid \rho_i x(p) \geqq h_i(p),\ p \in P \cap P_i\}.$$
$$= \{p \mid \rho_i \geqq \frac{h_i(p)}{x(p)},\ p \in P \cap P_i\}.$$

Now $\frac{h_i(p)}{x(p)}$ is i's minimum profitable market share.

From assumptions C 6 and E it follows, that $\frac{h_i(p)}{x(p)} > 1 \geqq \rho_i$ for small values of p and for large values of p (provided that $P_i \cap P \neq \emptyset$):

- if $p > c_i$ and $p \to c_i$, then by C 6, $h(p) \to \infty$; since $x(p)$ remains finite, $\frac{h_i(p)}{x(p)} \to \infty$
- if $p \geqq r$, then by assumption E, $px(p) < f_i(x(p))$ or equivalently, $h_i(p) > x(p)$.

By the continuity of $h_i(p)$ and $x(p)$ it follows that $P_i(\rho_i)$ is a compact set with $P_i(\rho_i) \subset P_i$. Note that $P_i(\rho_i)$ needs not be an interval and that it may be empty. Let i's (gross) profit function be

$$\pi_i(\rho_i, p) = p\rho_i x(p) - f_i(\rho_i x(p))$$

Then $\hat{p}(\rho_i)$ is i's most profitable price at share ρ_i, if

$$\pi_i(\rho_i, \hat{p}(\rho_i)) = \max \{\pi_i(\rho_i, p) \mid p \in P_i(\rho_i)\}$$

If i would have the right to fix a price, also binding for his competitors, he would fix this price. He then behaves as a monopolist facing the demand function $\rho_i x(p)$. Therefore we call $\hat{p}(\rho_i)$ also i's monopoly price at share ρ_i. If a maximum price $\bar{p}$ is given, then the set

$$P_i(\rho_i,\bar{p}) = P_i(\rho_i) \cap \{p \mid p \leqq \bar{p}\}$$

is the set of profitable priced permitted. We define $\hat{p}(\rho_i,\bar{p})$ as the most profitable price in $P_i(\rho_i,\bar{p})$

$$\pi_i(\rho_i,\hat{p}(\rho_i,\bar{p})) = \max\ \{\pi_i(\rho_i,p) \mid p \in P_i(\rho_i,\bar{p})\}$$

We call $\hat{p}(\rho_i,\bar{p})$ i's <u>restricted monopoly</u> price at share ρ_i and price restriction $\bar{p}$ because a firm facing the demand function $\rho_i x(p)$ and the maximum price $\bar{p}$, would fix this price. Prices $\hat{p}(\rho_i)$ and $\hat{p}(\rho_i,\bar{p})$ respectively, exist if and only if the sets $P_i(\rho_i)$ and $P_i(\rho_i,\bar{p})$ are non empty. (By assumption D 4, the profit function is bounded above). Obviously we have, if the monopoly prices exist,

$$c_i < \hat{p}(\rho_i,\bar{p}) \leqq \hat{p}(\rho_i) < r$$

The monopoly price $\hat{p}(\rho_i)$ needs not be unique.
Besides this the profit function $\pi_i(\rho_i,p)$ could have different <u>local</u> maxima.
Assume however

(F) In the interval $\rho_i^1 \leqq \rho_i \leqq \rho_i^2$, $P_i(\rho_i) \neq \emptyset$ and the profit function π_i has a single maximum $\hat{p}(\rho_i)$ and $\frac{\partial \pi_i}{\partial p} > 0$ for $p < \hat{p}(\rho_i)$ and $\frac{\partial \pi_i}{\partial p} < 0$ for $p > \hat{p}(\rho_i)$

<u>PROPOSITION</u>

If (F) holds, then $\rho_i^1 \leq \bar{\bar{\rho}}_i < \bar{\rho}_i \leq \rho_i^2$ implies $\hat{p}(\bar{\bar{\rho}}_i) > \hat{p}(\bar{\rho}_i)$

<u>Proof</u>

Profits as a function of total output are
$\pi_i(\rho_i,p(x)) = \rho_i p(x)x - f(\rho_i x)$

The maximum is attained for $\frac{\partial \pi_i}{\partial y} = 0$, hence

$$\rho_i(p'(x)x + p(x) - f'(\rho_i x)) = 0$$

$$\bar{\bar{\rho}}_i < \bar{\rho}_i \Rightarrow f'(\bar{\bar{\rho}}_i x) > f'(\bar{\rho}_i x), \text{ or}$$

$$p'x + p - f'(\bar{\bar{\rho}}x) < p'x + p - f'(\bar{\rho}_i x), \text{ hence}$$

$$\text{if } p'(\bar{\bar{x}})\bar{\bar{x}} + p(\bar{\bar{x}}) - f'(\bar{\bar{\rho}}\bar{\bar{x}}) = 0 = p'(\bar{x})\bar{x} + p(\bar{x}) - f'(\overline{\rho x})$$
$$\text{then } \bar{\bar{x}} < \bar{x}, \text{ hence } p(\bar{\bar{x}}) > p(\bar{x}).$$

So the monopoly price decreases if the market share increases. Provided that assumption (F) holds there, it will be lowest if $\rho_i = 1$.

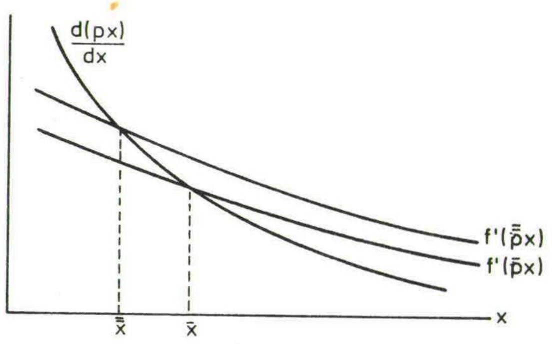


Figure 5

9. MARKET EQUILIBRIUM WITHOUT SELLING COSTS.

The market is defined by

- the set N of producers
- each producers cost function $f_i(y_i)$
- the total demand function $x(p)$
- the market share distribution $\rho_i(a)$

To each producer the prevailing market price $\tilde{p}$ and the activity indices $\tilde{a}_i$ of the others, are given.

The producer is maximizing profits and he has to decide on two things:

1) participate in the market or not

2) accept the prevailing price or lower it.

The profit function of firm i is $\pi_i(\rho_i(a),p) \equiv$

$$\pi_i(a,p) = p\rho_i(a)x(p) - f_i(\rho_i(a)x(p))$$

<u>Definition</u> 9.1

An <u>individual optimum</u> for firm i, given $\tilde{p}$ and $\tilde{a}_i$, is a price $\bar{p} \leqq \tilde{p}$ and $\bar{a}_i \in \{0,1\}$, such that

$$\pi_i(\bar{a}_i,\tilde{a}_i,\bar{p}) \geqq \pi_i(a_i,\tilde{a}_i,p)$$

for all $a_i \in \{0,1\}$ and $p \leqq \tilde{p}$.

Firm i will choose $a_i = 1$, if for some $p \leqq \tilde{p}$, $\pi_i(1,\tilde{a}_i,p) > 0$, or equivalently, if int $P_i(\rho_i,\tilde{p}) \neq \emptyset$. In this case the individual optimum will be the pair $(1,p)$, where p is the <u>restricted monopoly price</u> $\hat{p}(\rho_i(1,\tilde{a}_i),\tilde{p})$.
Firm i will choose $a_i = 0$, if for all $p \leqq \tilde{p}$, $\pi_i(1,\tilde{a}_i,p) < 0$, or equivalently if $P_i(\rho_i,\tilde{p}) = \emptyset$. The optimum solutions are then all pairs (o,p), such that $p \leqq \tilde{p}$. If his best result at $a_i = 1$ is zero, then both the solutions $(1,\hat{p}_i(\rho_i(1,\tilde{a}_i),\tilde{p}))$ and (o,p) for $p \leqq \tilde{p}$ are individual optima.

<u>Definition</u> 9.2

For the market, a <u>feasible solution</u> is a pair (a,p) such that

$$\forall i \in N: \pi_i(a,p) \geqq 0 \text{ and } a \neq 0$$

or equivalently, $\forall i \in N: a_i = 1 \Rightarrow \rho_i(a)x(p) \geqq h_i(p)$

<u>Definition</u> 9.3

A pair $(\bar{a},\bar{p})$ is a <u>Nash equilibrium</u> if $(\bar{a},\bar{p})$ is feasible and if $(\bar{a}_i,\bar{p})$ is an individual optimum for each i, given $\tilde{\bar{a}}_i$ and

$\tilde{p} = \bar{p}$

So in a Nash equilibrium we have

$$\forall i: \pi_i(\bar{a}_i, \bar{\tilde{a}}_i, \bar{p}) \geqq \pi_i(a_i, \bar{\tilde{a}}_i, p) \text{ for all } a_i \in \{0,1\},\ p \leqq \bar{p}.$$

Therefore for all active firms, the price p must be the restricted monopoly price for share $\rho_i(\bar{a})$ and price restriction $\bar{p}$: $\bar{p} = \hat{p}(\rho_i(\bar{a}), \bar{p})$.
This means, if $\bar{a}_i = 1$,

$$\pi_i(1, \bar{\tilde{a}}_i, \bar{p}) = \max\{\pi_i(1, \bar{\tilde{a}}_i, p) \mid p \leqq \bar{p}\} \geqq 0$$

which implies

$$\rho_i(1, \bar{\tilde{a}}) x(\bar{p}) \geqq h_i(\bar{p})$$

and if $\bar{a}_i = 0$

$$\pi_i(1, \bar{\tilde{a}}_i, p) \leqq 0 \text{ for all } p \leqq \bar{p}$$

so for all $p \leqq \bar{p}$: $\rho_i(1, \bar{\tilde{a}}_i) x(p) \leqq h_i(p)$.
So in a Nash equilibrium no active producer has an incentive to sleep in or to lower the price and no sleeping producer has an incentive to enter the market.
There are different Nash equilibria, with different sets of active producers and different prices. One solution certainly exists if the set of feasible solutions is not empty: the monopolistic solution where there is only one active firm.

Proposition 9.4

If the set of feasible solutions is not empty, then there exists at least one N.E.

Proof Since the set of feasible solutions is not empty, for some i, there exist ρ_i, such that $P_i(\rho_i) \neq \emptyset$ and so also $P_i(1) \neq \emptyset$. Let $A = \{i \mid P_i(1) \neq \emptyset\}$.

Choose $\bar{p} = \min \{q | q \in \bigcup_{A} P_i(1)\}$ and let $\bar{p} \in P_{i_o}(1)$.
Then the solution $(\bar{a},\bar{p})$ where $\bar{a}_{i_o} = 1$ and
$\bar{a}_i = 0$ if $i \neq i_o$, is a N.E.: $\pi_{i_o}(1,0,\bar{p}) = 0$,
$\bar{p} = \hat{p}(1,\bar{p})$, whereas $\pi_i(1,\tilde{\bar{a}},p) \leqq 0$ for any $p \leqq \bar{p}$

This needs not be the only solution with i_o as the only active producer. There may exist prices $\bar{\bar{p}} > \bar{p}$, such that $(\bar{a},\bar{\bar{p}})$ is an equilibrium (with $a_{i_o} = 1$ and $\bar{a}_j = 0$ if $j \neq i_o$).
Being a monopolist, firm i_o could increase his price so much that the attracts outsiders, i.e. such that for some i_1, $\pi_1(1,\tilde{\bar{a}}_1,p) > 0$ for some $p < \bar{\bar{p}}$. Similary if $(\bar{a},\bar{p})$ is an equilibrium with two active firms, $(\bar{a},\bar{\bar{p}})$ with $\bar{\bar{p}} > \bar{p}$ may also be an equilibrium, whereas at prices $p > \bar{\bar{p}}$ new firms will be attracted. Etc.
In the next section we shall explore the set of solutions for a particular case.

Before we do this, we first consider the case where <u>prices</u> are <u>fixed</u> from the outside. Then the firm has only to decide if it will produce or not, but it cannot lower the price.
Now a feasible solution is a vector a, such that $\pi_i(\bar{a}_i,\tilde{\bar{a}}_i,\bar{p}) \geqq 0$ for all i. A feasible solution $\bar{a}$ is a N.E. if

$$\pi_i(1,\tilde{\bar{a}}_i) > 0 \Rightarrow a_i = 1$$

Obviously the set of feasible solutions at a given price is a subset of the total set of feasible solutions. If $(\bar{a},\bar{p})$ is a N.F. at free prices, then $\bar{a}$ is also a N.E. for the price $\bar{p}$ fixed, but not conversely if $\bar{a}$ is a N.E. at fixed price $\bar{p}$, it remains possible, that for some i and for some $p < \bar{p}$

$$\pi_i(1,\tilde{\bar{a}}_i,p) > \pi_i(\bar{a},\bar{p})$$

Such a fixed price could be determined by the government, or by suppliers (resale price maintenance). Suppose however, that the active producers have collectively the right to determine a (minimum) price. Then it might occur that $(\bar{a},\bar{p})$ is a feasible solution, where $\bar{p}$ is not higher than the lowest monopoly price of the active firms at shares $\rho_i(\bar{a})$. So no active firm is interested in a price decrease. However it is possible that no sleeping firm could profitably enter the market at $\bar{p}$, but some firm could at a price $p < \bar{p}$, hence for the sleeping firm j

$$\pi_j(1,\tilde{\bar{a}}_j,\bar{p}) < 0 \text{ and } \pi_j(1,\tilde{\bar{a}}_j,p) > 0$$

(in example 1 below this is true for prices larger than 7,3).

REMARK

However each producer faces a decreasing demand function, and is a "monopolist" in that sense, he is not assumed to behave as a monopolist or oligopolist in any other sense. So he is not assumed to take into account reactions of his competitors on his own decisions.
Consider the following case: $(\bar{a},\bar{p})$ is a N.E. Hence for firm i, profits are not higher at any $p < \bar{p}$, given i's share $\rho_i(\bar{a})$.
At some price $\bar{\bar{p}} < \bar{p}$, firm j would leave the market and at the new a, firm i's profit would be higher.
In the present model firm i does not consider this possibility.
Also all kinds of cooperative behaviour are excluded.

10. IDENTICAL PRODUCERS.

Assume that all firms have the same cost function $f(y_i)$ and that the market share distribution is given by $\rho(a) = \frac{1}{\Sigma a_i}$, i.e. all active producers have the same market share.
In any feasible solution we should have $p \in P_i(\frac{1}{\Sigma a_i})$, hence $\frac{1}{\Sigma a_i}\, x(p) = \rho(a)x(p) \geqq h(p)$.
Hence $\Sigma a_i \leqq \frac{x(p)}{h(p)}$ and $\frac{x(p)}{h(p)}$ is the maximum number of firms that might operate in the market at no loss (where $h(p) \equiv h_i(p)$).
We define the function $\mu: P \cap P_i \rightarrow R$, where P and P_i are as defined in section 8 (P_i is identical for all firms)

$$\mu(p) = \frac{x(p)}{h(p)}$$

Since $\frac{1}{\mu(p)} = \frac{h(p)}{x(p)}$ is i's minimum profitable market share, $\mu(p)$ is <u>smaller</u> than 1 for small and for large values of p, as was shown in section 8; the set

$$\{p | \mu(p) \geqq 1 \text{ and } p \in P \cap P_i\} = P_i(1)$$

certainly contains a global maximum

$$\check{\mu}(p) = \max \{\mu(p) | p \in P_i(1) \quad = \mu(\check{p})\}$$

<u>Case I:</u> p is fixed from the outside.
In this case the solution is very simple. Let m be the number of active firms $m = \Sigma a_i$, so m is a positive integer.

<u>Proposition</u> 10.1

$\bar{a}$ is a Nash equilibrium for fixed price $\bar{p}$ if and only if, for $m = \Sigma a_i$,

$$m \leqq \min \{\mu(\bar{p}), n\} \leqq m + 1$$

This result is obvious. The number of active firms cannot exceed the total number of firms n. The maximum number of firms that may operate without loss is not larger than $\mu(\bar{p})$, whereas, if $m + 1 < \mu(\bar{p})$, then another firm could profitably enter the market.
If $m < \mu(\bar{p})$, then all firms make some profit, since in that case $\frac{1}{m}\, \bar{p}x(\bar{p}) - f(\frac{1}{m}\, x(\bar{p})) > 0$.
If $m = \mu(\bar{p})$, then all firms make zero profits.
If $\mu(\bar{p})$ is an integer, then there are two solutions: $m = \mu(\bar{p})$ with zero profits for all, and $m = \mu(\bar{p}) - 1$ with some profit for all active firms. For all integers $1 \leqq m \leqq \check{\mu}$, there exists a price such that there is "room" in the market for exactly that number of firms.

Case II: free prices

In this case each firm has the possibility to decrease the price. For $(\bar{a},\bar{p})$ to be a N.E., it is required

1) As in case I, each firm makes a non zero profit and no firm can profitably enter the market, hence

$$m \leqq \min \{\mu(\bar{p}),n\} \leqq m + 1$$

2) No active firm can raise his profit by a price decrease, hence

$$\text{(i) for } p \leqq \bar{p}\text{: } \pi(\tfrac{1}{m}\, x(p)) \leqq \pi(\tfrac{1}{m}\, x(\bar{p}))$$

This particularly implies, because of differentiability of the profit function:

$$\frac{d\pi_i}{dp} \leqq 0$$

which ensures that p is a local optimum.

In the profit function has a single maximum, then this is also a sufficient condition for (i) to be fullfilled.

3) No sleeping producer can profitably enter the market at a price $p < \bar{p}$. This requires that no $p < \bar{p}$ exists, such that $\mu(p) > m + 1$. From this it directly follows, that no $\bar{p}$ can be an equilibrium price if there exists $p < \bar{p}$ with $\mu(p) > \mu(\bar{p}) + 1$.

So we have

Proposition 10.2

A pair $(\bar{a},\bar{p})$ is a N.E., if and only if, for $m = \Sigma\bar{a}_i$,

(1) $m \leqq \min \{\mu(\bar{p}),n) \leqq m + 1$

(2) $\bar{p} = \hat{p}(\frac{1}{m},\bar{p})$, the restricted monopoly price at share $\frac{1}{m}$ and price restriction $\bar{p}$

(3) for all $p < \bar{p}$: $m + 1 > \mu(p)$

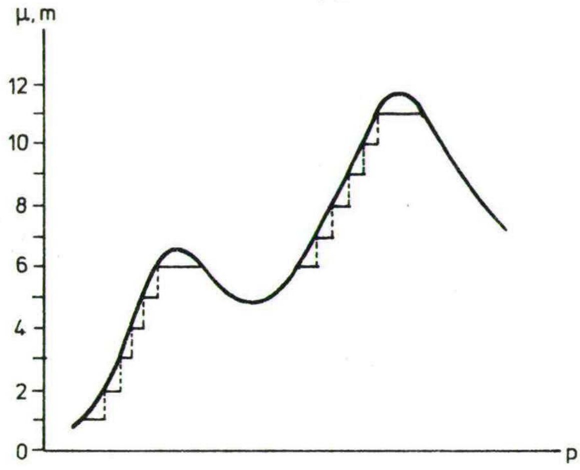


Figure 6

Property 10.3

Let $\check{\mu} = \max \mu(p)$. Then for all $a \in \{0,1\}^n$, for which $\Sigma a_i \leqq \check{\mu}$, there exist p such that (a,p) is a N.E.

Proof: Let $\check{m}$ be the largest whole number such that $\check{m} \leqq \check{\mu}$.
Define $T_k = \{p | \mu(p) \geqq k\}$ for $k = 1,2,\ldots,\check{m}$.
Then $T_1 \supset T_2 \supset \ldots \supset T_{\check{m}}$, and the set of feasible solutions is $F = \{(a,p) | \Sigma a_i = k,\ p \in T_k,\ k = 1,2,\ldots,\check{m}\}$
Define $S_k = \{p | p \in T_k,\ q < p \Rightarrow \mu(q) \leqq k+1,\ p \leqq \hat{p}(\frac{1}{k})\}$
Now

$$E = \{(a,p) | \Sigma a_i = k,\ p \in S_k \text{ and } k = 1,2,\ldots,\check{m}\}$$

it fullfills 1 of property 3.2.1, since $\Sigma a_i = k \leqq \mu(p)$; it fullfills 2, because p is the restricted monopoly price at share $\frac{1}{k}$ and restriction p and it fullfills 3, because for no $q \leqq p$, $\mu(q) > k + 1$.

EXAMPLE 1

Identical firms; no selling activity;
demand function: $x = 10 - p$;
cost function: $f(y_i) = \sqrt{8y_i}$;

hence $h_i(p) = \frac{8}{p^2}$ and $\mu(p) = \frac{(10-p)p^2}{8}$;

marginal profits $\pi_i'(p) = \frac{1}{m}\left(10 - 2p + \sqrt{\frac{2m}{10-p}}\right)$;

Fixed prices: for $p < 0{,}9$, $\mu(p) < 1$. Hence for $0{,}9 < p < 10$, m is the smallest whole number below $\mu(p)$.
The number of firms first increases with p and then decreases as is depicted in fig. 7. Some of the solutions are given in table I.

Free prices: The set of solutions is

$$\{m,p \mid 0{,}9 < p < 6{,}64 \text{ and } m \leqq \mu(p) \leqq m + 1\}$$

For $p > 6{,}64$, $\pi'(p) < 0$ (i.e. the effect of a price increase is negative, hence the effect of a price decrease is positive).
Figure 7 gives the number of firms in relation to the prices.
Some of the solutions are given in table I.

TABLE I

p	x	$\mu(p)$	m	y_i	π_i'
1	9	$1\frac{1}{8}$	1	9	> 0
2	8	4	4	2	> 0
3	7	$7\frac{7}{8}$	7	1	> 0
4	6	12	12	$\frac{1}{2}$	> 0
5	5	$15\frac{5}{8}$	15	$\frac{1}{3}$	> 0
6	4	18	18	$\frac{2}{9}$	> 0
6,64	3,36	18,51	18	0,19	≈ 0
7	3	$18\frac{3}{8}$	18	$\frac{3}{18}$	< 0
8	2	16	16	$\frac{1}{8}$	< 0
9	1	$10\frac{1}{8}$	10	$\frac{1}{10}$	< 0

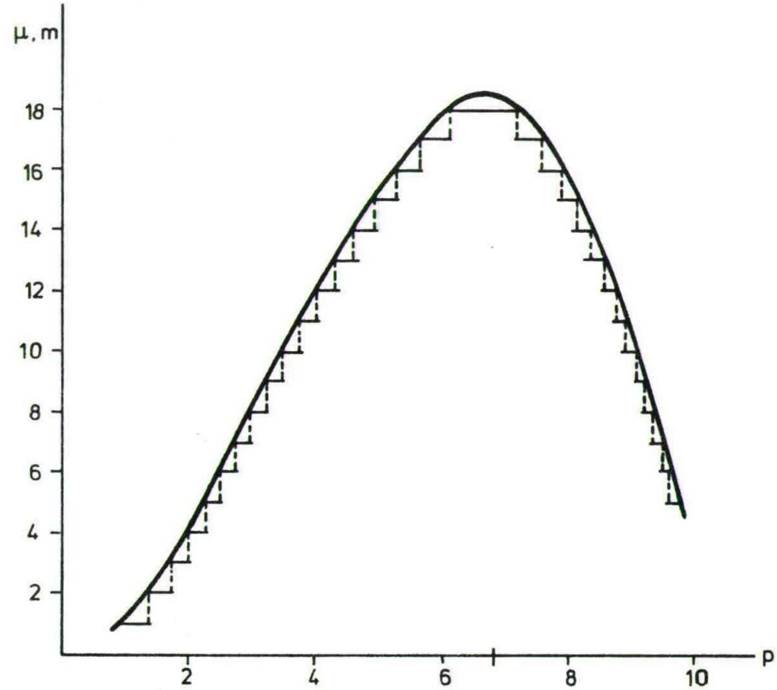


Figure 7

11. MARKET EQUILIBRIUM WITH SELLING COSTS.

The market is defined by

- The set of producers $N = \{1,2,\ldots,n\}$
- Each producer's cost function $f_i(y)$
- The total demand function $x(p)$
- The market share distribution $\rho_i(v)$

To each producer the prevailing market price $\tilde{p}$ is given, and the activity indices $\tilde{a}_i$ and the selling expenses $\tilde{s}_i$ of the others, are given. (Remember that $v_i = (a_i, s_i)$ and $v = (v_i, \tilde{v}_i)$). The producer is maximizing his profit and has to decide on three things:

1) participate in the market or not: a_i is 0 or 1;
2) fix the amount of selling costs s_i;
3) accept the prevailing market price or lower it to its optimal value.

Let the producers gross profits be

$$\pi_i(a_i,s_i,\tilde{v}_i,p) = p\rho_i(a_i,s_i,\tilde{v}_i)x(p) - f_i(\rho_i(a_i,s_i,\tilde{v}_i)x(p))$$

whereas net profits are gross profits minus selling costs s_i, denoted by ϕ_i are

$$\phi_i(a_i,s_i,\tilde{v}_i,p) = \pi_i(a_i,s_i,\tilde{v}_i,p) - s_i$$

Definition 11.1

An individual optimum for firm i, given $\tilde{p}$, and $\tilde{v}_i$ is a price $\bar{p} \leqq \tilde{p}$, $\bar{s}_i > 0$ and $\bar{a}_i \in \{0,1\}$ such that

$$\phi_i(\bar{a}_i,\bar{s}_i,\tilde{v}_i,\bar{p}) = \max\ \{\phi_i(a_i,s_i,\tilde{v}_i,p)\,|\,a_i \in \{0,1\}\ s_i \geqq 0,\ p \leqq \tilde{p}\}$$

Note that if $\bar{a}_i = 0$, $\bar{s}_i = 0$ and $\bar{p}$ may have any value $\bar{p} \leqq \tilde{p}$.

Definition 11.2

A feasible solution for the market is a pair $(\bar{v},\bar{p})$ such that $\bar{a}_i = 1 \Rightarrow \phi_i(\bar{v},\bar{p}) \geqq 0$.

Definition 11.3

A Nash Equilibrium is a pair $(\bar{v},\bar{p})$ such that $(\bar{v},\bar{p})$ is feasible and $(\bar{a}_i,\bar{s}_i,\bar{p})$ is an individual optimum given $\tilde{\bar{v}}_i$ and $\bar{p}$.

So in a Nash Equilibrium, if $\bar{a}_i = 1$, ϕ_i attains a maximum at $\bar{s}_i$ and $\bar{p}$. If $\bar{a}_i = 0$, ϕ_i is nowhere strictly positive. At his market share $\rho_i = \rho_i(\bar{v})$, the price $\bar{p}$ is i's restricted monopoly price $\hat{p}(\rho_i,\bar{p})$: given ρ_i and hence s_i, both i's gross profits and his net profits are maximum at $\bar{p}$.

If the price is given from the outside, the firm has only to choose a_i and s_i. Then $\bar{v}$ is a N.E. for fixed $\bar{p}$ if, for each i, $\phi_i(\bar{v},\bar{p}) = \max\ \{\phi_i(a_i,s_i,\tilde{\bar{v}}_i,\bar{p})\,|\,a_i \in \{0,1\},\ s_i \geqq 0\}$.

Obviously a N.E. $(\bar{v},\bar{p})$ is also N.E. $(\bar{v})$ for $\bar{p}$ fixed. The converse is not true.
Gross profits can also be expressed in terms of the market share ρ_i, with the help of the selling cost function as defined in section 6. Replacing s_i by ρ_i in the profit function, we get

$$\phi_i(a_i,\rho_i,\tilde{v}_i,p) = a_i\left[p\rho_i x(p) - f_i(\rho_i x(p)) - g_i(\rho_i,\tilde{v}_i)\right]$$

This profit function is defined for ρ_i, such that

$$\rho_i^0(\tilde{v}_i) \leqq \rho_i \leqq \rho_i^{\sup}(\tilde{v}_i) \qquad \text{(see section 6)}$$

Let M be a real number such that $M > \max\limits_{p>0} px(p) = p^0x(p^0)$ by assumption D 4, and $\rho_i^M(\tilde{v}_i) = \rho_i(1,M,\tilde{v}_i) \leqq \rho_i^{\sup}(\tilde{v}_i)$ then we may require $\rho_i(\tilde{v}_i) \leqq \rho_i \leqq \rho_i^M(\tilde{v}_i)$: for larger values of ρ_i the selling costs are at least M, so profits are negative at $a_i = 1$. This cannot give an optimal solution.

In an individual optimum $(\bar{v}_i,\bar{p})$ given $(\tilde{v}_i,\tilde{p})$ we must have:

$$\phi_i(\bar{a}_i,\bar{\rho}_i,\tilde{v}_i,\bar{p}) = \max\ \{\phi_i(a_i,\rho_i,\tilde{v}_i,p)\}\,|\,a_i \in \{0,1\},$$
$$\rho_i^0(\tilde{v}_i) \leqq \rho_i \leqq \rho_i^M(\tilde{v}_i),\ p \leqq \tilde{p}\}$$

This maximum <u>always exists:</u>

The function

$$p\rho_i x(p) - f_i(\rho_i x(p)) - g_i(\rho_i,\tilde{v}_i)$$

is continuous on the compact set

$$\{(p,\rho_i)\,|\,0 \leqq p \leqq \tilde{p} \text{ and } \rho_i^0(\tilde{v}_i) \leqq \rho_i \leqq \rho_i^M(\tilde{v}_i)\}$$

so it has a maximum. If this maximum is <u>negative</u> than the profit function is maximum for $a_i = 0$. If the maximum is non-negative and attained at $(\bar{\rho}_i, \bar{p})$, then $(\bar{v}_i, \bar{p})$ is an individual optimum with $\bar{a}_i = 1$ and $\bar{s}_i = g_i(\bar{\rho}_i, \tilde{v}_i)$.

If $\bar{a}_i = 1$, we have in this optimum:

I the optimum price $\bar{p} = \hat{p}_i(\rho_i)$, i.e. the optimum price is the restricted monopoly price, as in the case of no selling costs. This is so because for $\bar{\rho}_i$ fixed, the marginal profit does not depend on the selling costs. Obviously the restricted monopoly price only gives an optimum if net profits exceed selling costs.

II For $\bar{p}$ fixed, the profit function attains its maximum at $\bar{\rho}_i$

(1) if $\bar{\rho}_i = \rho_i^M(\tilde{v}_i)$, then $\bar{s}_i = g_i(\bar{\rho}_i, \tilde{v}_i) < M$, since $\phi_i \geqq 0$; hence $\bar{s}_i$ is i's satiation level of selling costs at $\tilde{v}_i$;

(2) if $\bar{\rho}_i < \rho_i^M(\tilde{v}_i)$, then the total cost function $f_i + g_i$ must be locally concave with respect to ρ_i at $\bar{\rho}_i$. This type of solution certainly occurs if there does not exist a satiation level of selling costs.

By adding production costs and selling costs, and substituting $\frac{y_i}{x(p)}$ for ρ_i, we get the <u>total cost</u> function of sales:

$$k_i(y_i, \tilde{v}_i, p) = f_i(y_i) + g_i\left(\frac{y_i}{x(p)}, \tilde{v}_i\right)$$

(for $0 \leqq y_i \leqq \rho_i^M x(p)$).

At a given price p the optimum is attained at a point where the total cost function is locally convex.

A particularly interesting case occurs, if the total cost function is <u>convex</u> in some interval $\bar{\bar{y}}_i \leqq y_i \leqq \rho_i^M x(p)$. This could happen if the selling cost function is completely convex, or convex from a certain level of sales, and its convexity compensates the concavity of the production costs.

This requires that marginal selling costs are increasing and become more and more increasing (or equivalently, that the marginal effectiveness of selling costs is more and more decreasing). This seems a quite reasonable assumption. (see figure 8)

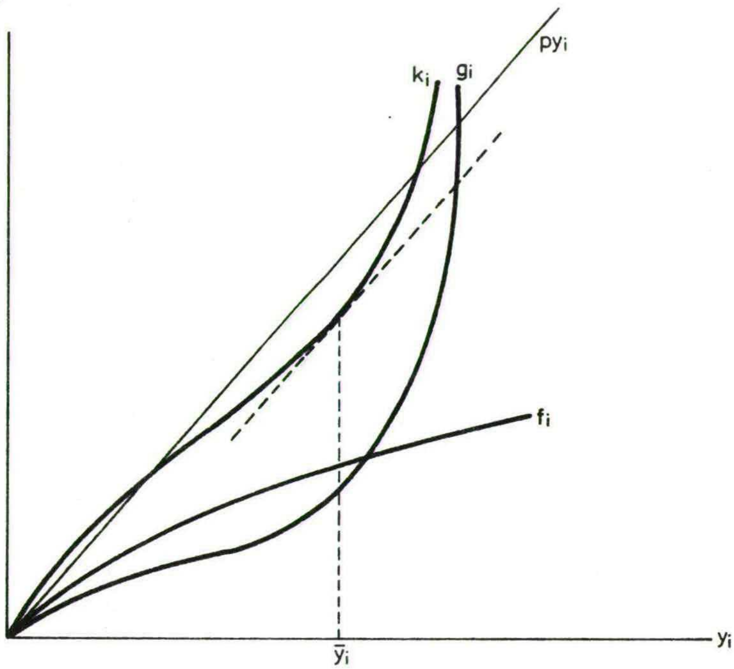


Figure 8

Then the total marginal and mean cost functions are "U-shaped", which brings us back to the traditional case however with price and decisions of other firms appearing in the cost function.

12. IDENTICAL PRODUCERS.

Assume that all firms have the same cost function f(y) and that the market share is proportional to each producer's selling costs (see section 6).

$$(12.1) \qquad \rho_i = \frac{s_i}{s_i+S_i} \quad \text{for } s_i \geqq 0 \text{ and } S_i = \sum_{j\neq i} s_j > 0$$

This function is twice differentiable and we have

$$(12.2) \qquad \frac{\partial \rho_i}{\partial s_i} = \frac{S_i}{(s_i+S_i)^2} > 0; \quad \frac{\partial^2 \rho_i}{\partial s_i^2} = \frac{-2S_i}{(s_i+S_i)^3} < 0$$

$$\frac{\partial \rho_i}{\partial s_j} = - \frac{s_i}{(s_i+S_i)^2} < 0; \quad \frac{\partial^2 \rho_i}{\partial s_j^2} = \frac{2s_i}{(s_i+S_i)^3} > 0$$

so the share is a concave function in s_i(for S_i fixed). The selling cost function becomes

$$(12.3) \qquad s_i = g_i(s_i, S_i, p) = \frac{y_i}{x(p)-y_i} S_i \text{ for } 0 \leqq y_i \leqq x(p)$$

and we have

$$\frac{\partial s_i}{\partial y_i} = \frac{x}{(x-y_i)^2} S_i > 0; \quad \frac{\partial^2 s_i}{\partial y_i^2} = \frac{2x}{(x-y_i)^3} S_i > 0$$

so the selling cost function is <u>strictly convex</u>.

Case I: <u>price fixed from the outside</u>

Gross profits are

$$(12.4) \qquad \phi_i = py_i - f(y_i) - \frac{y_i}{x-y_i} S_i \text{ for } 0 \leqq y_i \leqq y(p)$$

Production costs and selling costs are complementary. Maximization of ϕ_i gives

$$(12.5) \qquad p - f'(y_i) - \frac{x}{(x-y_i)^2} S_i = 0$$

If m is the number of active producers, and we consider solutions <u>identical</u> for all <u>active</u> producers, then $y = my_i$ and $S_i = (m-1)s_i$ hence

$$p - f'(y_i) - \frac{my_i}{(m-1)^2 y_i^2}(m-1)s_i = 0$$

or

$$(12.6) \qquad s_i = \frac{m-1}{m}(p-f'(y_i))$$

Because $\phi_i \geqq 0$

$$(12.7)\qquad py_i - f(y_i) - \frac{m-1}{m}(p-f'(y_i))y_i \geqq 0$$

or

$$\frac{1}{m}\left[py_i - f'(y_i)y_i - m\{f(y_i) - f'(y_i)y_i\}\right] \geqq 0$$

it follows

$$(12.8)\qquad m \leqq \frac{p-f'(y_i)}{\frac{f(y_i)}{y_i} - f'(y_i)} \quad \text{where } y_i = \frac{1}{m}x(p)$$

So (12.8) shows that the number of active producers will be equal to the largest whole number, which is smaller than the quotient of price minus marginal cost and mean cost minus marginal cost. Each firm's output equals y_i and total output is $x(p) = my_i$.

Each firm spends from his gross profits $py_i - f(y_i)$, the amount $\frac{m-1}{m}(p-f'(y_i))y_i$ on selling cost. So most of gross profits are spent on selling activity in the equilibrium solution.

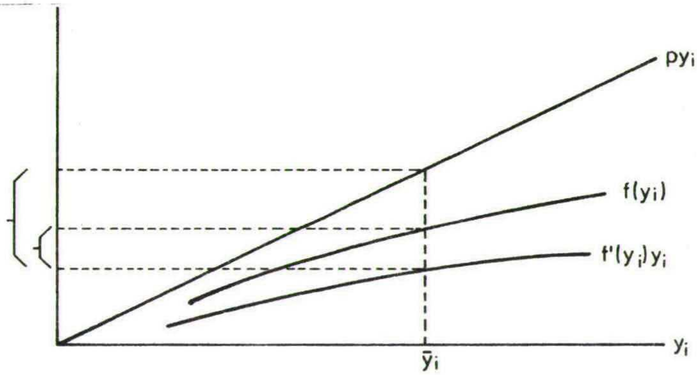


Figure 9

Remarks

1) The solution considered above is a symmetric one, all firms having the same output. However non-symmetric solutions are also possible as is shown by example 3.

2) The solution holds for p fixed. It will also be a solution for variable p, if $p = \hat{p}(\rho_i(\tilde{s}_i), p)$.

Case II: Free price

The optimum must fullfill two conditions.

(i) s_i must be such that it is optimal for S_i, given p

(ii) p must be such that it is optimum for given $\rho_i = \frac{s_i}{s_i+S_i}$

From $\rho_i = \frac{s_i}{s_i+S_i}$ it follows $s_i = \frac{\rho_i}{1-\rho_i}S_i$

The profit function is

$$(12.9) \qquad \phi(\rho_i, S_i, p) = py_i - f(y_i) - s_i$$

$$= p\rho_i x(p) - f(\rho_i x(p)) - \frac{\rho_i}{1-\rho_i}S_i$$

Differentiation with respect to ρ_i and p gives

$$(12.10) \qquad \frac{\partial \phi_i}{\partial \rho_i} = \left(p - f'(\rho_i x(p))\right) x - \frac{1}{(1-\rho)^2} S_i = 0$$

$$(12.11) \qquad \frac{\partial \phi_i}{\partial p} = \rho_i \{x + \quad p - f'(\rho_i x) \frac{dx}{dp}\} \geqq 0$$

($\geqq 0$, since p is only flexible below)

(12.11) is the formula to express a monopoly solution, so its solution must be the restricted monopoly price $\hat{p}(\rho_i, p) = p$. Instead of (12.10) and (12.11) we may write

$$(12.12) \qquad (1-\rho)^2 = \frac{S_i}{p - f'(\rho_i x) \ x} \quad \text{and } x + \quad p - f'(\rho_i x) \frac{dx}{dp} \geqq 0$$

As in the case of fixed prices, equilibrium selling costs are (in the symmetric solution):

$$s_i = \frac{m-1}{m}(p - f'(\tfrac{1}{m}x))y_i$$

The number of active firms is the largest whole number such that

$$m \leqq \frac{p-f'(\frac{1}{m}x)}{\frac{f(\frac{1}{m}x)}{\frac{1}{m}x} - f'(\frac{1}{m}x)}$$

with the condition that

$$\frac{dx}{dp} \geqq \frac{-x}{p-f'(\rho_i x)} = \frac{-x}{p-f'(\frac{1}{m}x)}$$

which implies that p is the restricted monopoly price.

EXAMPLE 2.

Identical firms; with selling activity; demand function: $x = A-Bx$, $A > 0$, $B > 0$; cost function: $f(y_i) = \alpha+\beta y_i$.

I Price fixed.

By (11.8)

$$m \leqq \frac{p-f'(y_i)}{\frac{f(y_i)}{y_i} - f'(y_i)} = \frac{p-\beta}{\frac{\alpha+\frac{\beta}{m}x}{\frac{1}{m}x} - \beta} = \frac{(p-\beta)x}{\alpha m}$$

hence

$$m^2 \leqq \frac{(p-\beta)x}{\alpha} \equiv \nu(p)$$

If we choose the parameters as indicated in table II, then m is at most 50, which number is attained at a price 500,5. For lower prices and for higher prices m is lower. If the price decreases to 10.5, the number of firms is equal to 9.

II Price variable.

At variable prices, a price decrease should not be profitable

$$\frac{\partial \phi_i}{\partial p} = \frac{1}{m}(px'+y-\beta x') = \frac{1}{m}(A-2Bp+\beta B) \geqq 0$$

This implies

$$p \leqq \frac{1}{2}(\frac{A}{B}+\beta)$$

If we choose the parameters as indicated in table II we have

$$p \leqq 500,5.$$

Note that in this example selling costs are very large in relation to production costs.

Table II

p	x	ν	m	y_i	py_i	f_i	s_i	$\frac{\partial \phi_i}{dp}$
700,5	300	2100	45	6,66	4665,33	103,33	4558,4	< 0
600,5	400	2400	48	8,33	5002,17	106,16	4893,8	< 0
500,5	500	2500	50	10	5005	105	4900	= 0
400,5	600	2400	48	12,5	5006,25	106,25	4898	> 0
300,5	700	2100	45	15,5	4657,75	107,5	4546	> 0
200,5	800	1600	40	20	4010	110	3900	> 0
100,5	900	900	30	30	3015	115	2900	> 0
50,5	950	475	21	45,3	2288	122,7	2157	> 0
20,5	980	196	14	70	1435	135	1300	> 0
10,5	990	99	9	110	1155	155	977	> 0

$A = 1000,5$; $B = 1$; $\alpha = 100$, $\beta = 1/2$

EXAMPLE 3.

Non symmetrical solution in the case of identical firms and selling expenses.

Demand function: $x = -\frac{45}{32}p + \frac{196}{32}$;

Cost function : $f_i(y_i) = \frac{y_i}{y_i+1}$;

By (12.8)

$$m \leqq \frac{p-\frac{1}{(1+y_i)^2}}{\frac{1}{1+y_i} - \frac{1}{(1+y_i)^2}} = \frac{p(1+y_i)^2-1}{y_i}$$

and after substituting $y_i = \frac{1}{m}x$, it follows

$$m \leqq \frac{px}{\sqrt{px+p}-p}$$

Choose $p = \frac{24}{10}$, then $x = \frac{11}{4}$ and we find $m \leqq 11$. For $m = 11$, we find, applying (12.6), $s_i = \frac{2}{5}$.
However, assume that there are 9 (type A) firms having output $\frac{1}{4}$, and 1 (type B) firm, having, output $\frac{1}{2}$; this also is an equilibrium, if $s_A = \frac{2}{5}$ and $s_B = \frac{4}{5}$: with $S_B = \frac{18}{5}$, equation (12.5) is fullfilled

$$p-f'(\tfrac{1}{2}) - \frac{x}{(x-\frac{1}{2})^2}S_B = 0$$

Now $\phi_A = 0$ and $\phi_B > 0$. This is also a solution for variable prices, since

$$\frac{\partial\phi_A}{\partial p} > 0 \text{ and } \frac{\partial\phi_B}{\partial p} = 0 \quad \text{at} \quad p = \frac{24}{10}$$

(Other solutions: 7 type A firms an 2 type B firms, etc.).

13. FINAL REMARKS.

1) The most interesting conclusion from our two models seems to be, that the solution is highly indeterminate, in the sense that many equilibria can occur and these are more or

less stable, once they exist.
In the no-selling costs case the number of firms is adjusted to the price. In the case of selling costs these are blown up, using a large part of gross profits, in such a way that each firm's total marginal costs become at least locally increasing.
Selling costs may serve to "convexify" the total cost function.

2) The appropriateness of the Nash Equilibrium as an equilibrium concept for the present models may be questioned. An extreme alternative would be to assume that only a single firm can survive, e.g. by a merger of all active firms. The monopolist of this case would have to set a price and make selling expenses so as to exclude profitable entry. This solution is one of the Nash equilibria considered. Further there is room for different forms of cooperation between both active and sleeping firms, e.g. cooperative price increases and cooperative reductions of selling expenses.

3) The case of fixed prices may be applied to the case of resale price maintenance. Let the firms be resalers, selling a single commodity or a basket of commodities, at prices determined by the producers. Suppose that the resalers have decreasing mean and marginal costs with respect to the quantity sold.
If the resalers have no selling costs, then our model tells us, that the number of firms will adapt to the fixed price, in such a way that nobody will make more then a small profit.
If there are possibilities to increase one's market share by selling activity then firms will "advertise away" most of their gross profits and also the number of firms will be adapted to the fixed prices. In both cases it is possible that prices are "too" high, i.e. either the fixed price is above the restricted monopoly price, or at free prices new firms will enter at lower prices.

4) Note that in the present paper equilibrium solutions are described, not the way in which an equilibrium is attained. There is an equilibrium if and only if no firm has an incentive to change the price or his selling expenses or to leave or enter the market. If there is no equilibrium, then an adjustment process will go on, which will end, if it ends at all, in an equilibrium. Where it ends depends on the starting position and on the character of the adjustment process.

   It is also possible that the way in which the adjustment process develops, gives rise to a change of the model, e.g. by government regulations or by the creation of cooperation.

5) A disequilibrium could originate from an equilibrium where a price increase of inputs occurs (inflation), which causes that some or all firms make a loss. Within the framework of our models, no firms can increase the price of the output. So as a consequence of price rigidity, some firms will have to leave the market, from which a new equilibrium may result. Of course it is also possible that a cooperative price increase is organized by all active firms; this may also result in a Nash equilibrium provided that the price is not increased so much as to provoke entry.

6) The assumption of decreasing mean costs at any level, may seem very strong. However if mean costs start increasing at a very high level, higher than total demand, this certainly will give the same result.
   It seems however that the theory developed in this paper is also applicable if the cost function is a traditional one. This will be a subject of further research.

G.W.

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